Calculus 1 Test 2 — Summer 2013

NAME KEY STUDENT NUMBER

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Use equal signs when things are equal (and don't use them when things are not equal). If applicable, put your answer in the box provided. Each numbered problem is worth 10 points. You may use your calculator any way you see fit, but **you may not share calculators**!!! Use my square bracket notation when applicable.

1. (a) Differentiate $y = \left(\frac{\sqrt{x}}{1+x}\right)^2$. Use my square bracket notation and do not simplify.

p213 #31

$$y' = 2\left(\frac{\sqrt{x}}{1+x}\right) \frac{[\frac{1}{2}x^{-1/2}](1+x) - (\sqrt{x})[1]}{(1+x)^2}$$

p152 #10

(b) A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec reaches a height of $s = 24t - 0.8t^2$ m in t sec. Find the rock's velocity and acceleration at time t.

velocity $v(t) = \frac{ds}{dt} = 24 - 1.6t$ m/sec

acceleration $a(t) = \frac{ds}{dt} = -1.6$ m/sec^2

$\boxed{v(t) = 24 - 1.6t \text{ m/sec} \quad a(t) = -1.6 \text{ m/sec}^2}$

2. (a) Find y'' if $y = \sec x$. Use my square bracket notation and do not simplify.

p160 #33b

$$y' = \sec(x)\tan(x)$$

$$y'' = [\sec(x)\tan(x)](\tan(x)) + (\sec(x))[\sec^2(x)]$$

p 168 #58

(b) Differentiate $y = \left(e^{\sin(t/2)}\right)^3$. Use my square bracket notation and do not simplify.

$$y' = 3\left(e^{\sin(t/2)}\right)^2\left[e^{\sin(t/2)}\left[\cos\left(\tfrac{t}{2}\right)\left[\tfrac{1}{2}\right]\right]\right]$$

3. Consider $x \sin 2y = y \cos 2x$. Find the slope of the line tangent to the graph of this equation at the point $(\pi/4, \pi/2)$.

p174 #38

well, $\frac{d}{dx}[x \sin(2y)] = \frac{d}{dx}[y \cos(2x)]$

$$[1](\sin(2y)) + (x)[\cos(2y)[2y']] = [y'](\cos(2x)) + y[-\sin(2x)[2]],$$

$$2xy'\cos(2y) - y'\cos(2x) = -\sin(2y) - 2y\sin(2x)$$

$$y' = \frac{-\sin(2y) - 2y\sin(2x)}{2x\cos(2y) - \cos(2x)}, \quad \text{so for } (x,y) = \left(\tfrac{\pi}{4}, \tfrac{\pi}{2}\right),$$

$$y' = \frac{-\sin(\pi) - \pi\sin(\pi/2)}{\frac{\pi}{2}\cos(\pi) - \cos\left(\frac{\pi}{2}\right)} = \frac{-\pi}{-\frac{\pi}{2}} = 2$$

$$\boxed{2}$$

p185 #94

4. (a) Differentiate $y = x^{\sin x}$. HINT: Use logarithmic differentiation.

well, $\ln(y) = \ln\left(x^{\sin x}\right) = \sin(x)\cdot\ln x$

$$\Rightarrow \frac{d}{dx}[\ln(y)] = \frac{d}{dx}[\sin(x)\cdot\ln x]$$

$$\frac{1}{y}[y'] = [\cos(x)](\ln x) + (\sin(x))\left[\frac{1}{x}\right]$$

$$\Rightarrow y' = y\left(\cos(x)(\ln x) + \frac{\sin x}{x}\right) \Rightarrow \boxed{y' = x^{\sin x}\left(\cos(x)\ln x + \frac{\sin x}{x}\right)}$$

p214 #60

(b) Differentiate $y = (1+t^2)\cot^{-1} 2t$. Use my square bracket notation and do not simplify.

$$y' = [2t]\left(\cot^{-1}(2t)\right) + (1+t^2)\left[\frac{-1}{1+(2t)^2}[2]\right]$$

p199 #31

5,6. A spherical balloon is inflated with helium at the rate of 100π $\mathrm{ft}^3/\mathrm{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 ft? HINT: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

(1)

100π ft^3/min

(2) We know $\frac{dV}{dt} = 100\pi$ ft^3/min.

(3) The question is $\frac{dr}{dt} = ?$ when $r = 5$ ft.

(4) We know $V = \frac{4}{3}\pi r^3$.

(5) Differentiating implicitly with respect to time:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \left[\frac{dr}{dt}\right]$$

or $$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

(6) WHEN $r = 5$ ft we have

$$\frac{dr}{dt} = \frac{1}{4\pi(5\,ft)^2}\left(100\pi\ ft^3/min\right)$$

$$= \frac{1}{100\pi}(100\pi)\ ft/min = 1\ ft/min$$

$\boxed{1\ ft/min}$

p228 #29

7. Find the maxima and minima of $g(x) = \sqrt{4-x^2}$ on the interval $[-2, 1]$. Explain your answer.

Well, $g'(x) = \frac{1}{2}(4-x^2)^{-1/2}[-2x] = \frac{-x}{\sqrt{4-x^2}}$, so the critical points are $x=0$ where g' is 0, and $x = \pm 2$ where g' is undefined. Consider:

x	$g(x)$
-2	0
0	2
-1	$\sqrt{3}$

Since g is continuous on $[-2,1]$:
g has a MAX of 2 at $x=0$,
g has a MIN of 0 at $x=-2$.

$$\boxed{\begin{array}{lll} \text{MAX} & 2 & \text{at } x=0 \\ \text{MIN} & 0 & \text{at } x=-2 \end{array}}$$

8. (a) State the Mean Value Theorem with hypotheses and conclusion.

p231

Suppose $y=f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on (a,b). Then there is at least one point $c \in (a,b)$ at which

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

p236 #4

(b) Explain why $f(x) = \sqrt{x-1}$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 3]$. Find the point $c \in [1, 3]$ which is guaranteed to exist by the conclusion of the Mean Value Theorem.

$f(x) = \sqrt{x-1}$ is continuous on $[1, \infty)$ and so is continuous on $[1,3]$,
$f'(x) = \frac{1}{2}(x-1)^{-1/2} = \frac{1}{2\sqrt{x-1}}$ is defined on $(1,\infty)$ and so f
is differentiable on $(1,3)$. Next, set $f'(c) = \frac{f(b)-f(a)}{b-a}$:

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{(3)-1} - \sqrt{(1)-1}}{(3)-(1)} = \frac{\sqrt{2}}{2} \Rightarrow \frac{1}{4(c-1)} = \frac{2}{4}$$

$$\Rightarrow c-1 = \frac{1}{2} \Rightarrow c = \frac{1}{2}+1 = \frac{3}{2} \in [1,3] \qquad \boxed{c = \frac{3}{2}}$$

p252 #33

9, 10. Consider $y = f(x) = 2x - 3x^{2/3}$. Find the critical points, intervals on which f is INC/DEC, points of inflection, intervals on which f is CU/CD, extrema, and graph.

Well, $f'(x) = 2 - 3\left[\frac{2}{3} x^{-1/3}\right] = 2 - 2x^{-1/3} = 2 - \frac{2}{\sqrt[3]{x}}$.

So $x = 1$ is a critical point since $f'(1) = 0$.
$x = 0$ is a critical point since $f'(0)$ is undefined.

Consider

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
k	-1	$1/8$	8
$f'(k)$	4	-2	1
$f'(x)$	$+$	$-$	$+$
$f(x)$	INC	DEC	INC

f is INC on $(-\infty, 0) \cup (1, \infty)$
f is DEC on $(0, 1)$

f has a MAX at $x = 0$ of $f(0) = 0$
f has a MIN at $x = 1$ of $f(1) = -1$

Next, $f''(x) = -2\left[\frac{1}{3} x^{-4/3}\right] = \frac{2}{3x^{4/3}}$, so $x = 0$ is a potential point of inflection.

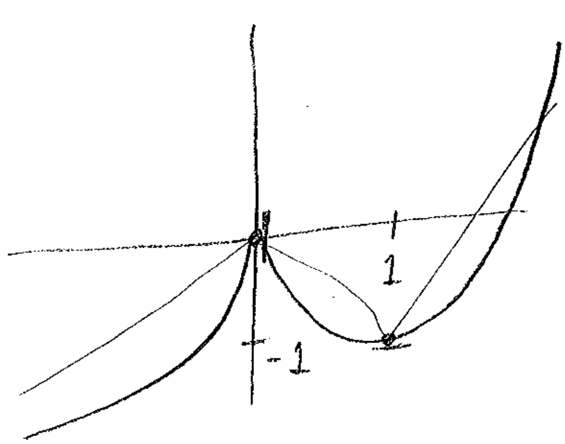
Consider

	$(-\infty, 0)$	$(0, \infty)$
k	-1	1
$f''(k)$	$2/3$	$2/3$
$f''(x)$	$+$	$+$
$f(x)$	CU	CU

f is CU on $(-\infty, \infty)$

Notice that f has NO points of inflection.

so



p252 #58

Bonus 1. Consider $y = \dfrac{e^x}{1+e^x}$. Find the horizontal asymptotes, the critical points, intervals on which f is INC/DEC, points of inflection, intervals on which f is CU/CD, extrema, and graph.

We know $\lim_{x \to +\infty} (e^x) = \infty$ and $\lim_{x \to -\infty} (e^x) = 0$. So

$$\lim_{x \to -\infty} \frac{e^x}{1+e^x} = \frac{0}{1+0} = 0 \text{ and } \lim_{x \to +\infty} \left(\frac{e^x}{1+e^x}\right) = \lim_{x \to +\infty} \left(\frac{e^x}{1+e^x}\right)\left(\frac{e^{-x}}{e^{-x}}\right)$$

$$= \lim_{x \to +\infty} \left(\frac{1}{e^{-x}+1}\right) = \frac{1}{0+1} = 1.$$

Next, $f'(x) = \dfrac{[e^x](1+e^x) - (e^x)[e^x]}{(1+e^x)^2} = \dfrac{e^x}{(1+e^x)^2}$. So

$f'(x) > 0$ for all x and $\boxed{f \text{ is INC on } (-\infty, \infty)}$.

Now $f''(x) = \dfrac{[e^x](1+e^x)^2 - (e^x)[2(1+e^x)[e^x]]}{(1+e^x)^2}$

$$= \frac{e^x(1+e^x)\left((1+e^x) - 2e^x\right)}{(1+e^x)^2} = \frac{e^x(1-e^x)}{(1+e^x)}.$$

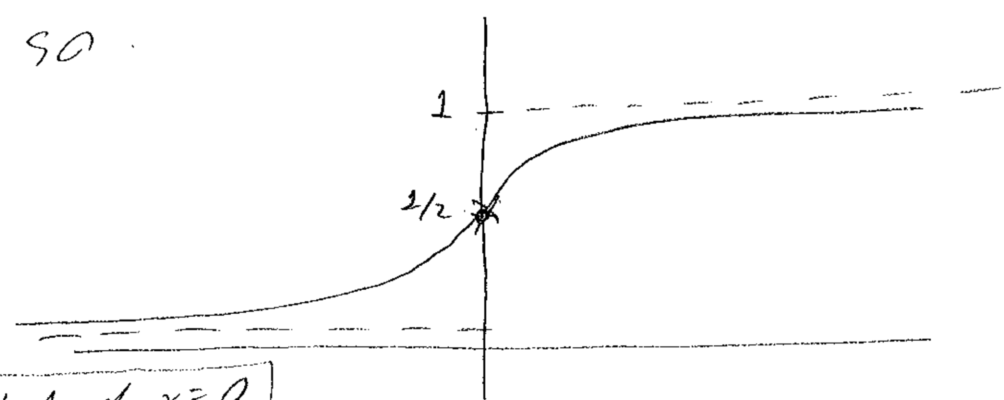
So $x = 0$ is a potential point of inflection. Consider

	$(-\infty, 0)$	$(0, \infty)$
h	-1	1
$f'(h)$	$\frac{(+)(+)}{(+)}$	$\frac{(+)(-)}{(+)}$
$f'(x)$	$+$	$-$
$f(x)$	CU	CD

So f is CU on $(-\infty, 0)$
f is CD on $(0, \infty)$.

$x = 0$ IS a point of inflection

So.



H.A. of $x = 1$

f has no MAX, no MIN.

H.A. of $x = 0$